

ON THE EXISTENCE OF GEOMETRIC MODELS FOR FUNCTION FIELDS IN SEVERAL VARIABLES

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ABSTRACT. In this paper we will give an explicit construction of the geometric model for a prescribed Galois extension of a function field in several variables over a number field. As a by-product, we will also prove the existence of quasi-galois closed covers of arithmetic schemes, which is a generalization of the pseudo-galois covers of arithmetic varieties in the sense of Suslin-Voevodsky.

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INTRODUCTION

Let F be a number field and let E be a finitely generated extension of F . If $\text{tr.deg}_F E = 1$, this is on the theory of function fields of one variable, especially on the Riemann-Roch Theory.

Consider the case that $[E : F] < \infty$. In recent decades one has been attempted to use the related data of arithmetic varieties X/Y to describe such a given (Galois) extension E/F (for example, see [3, 7, 8, 9, 11, 12, 13, 15, 16]). The reason is that there is a nice relationship between them:

For the case that $\dim X = \dim Y$, as it has been seen, under certain conditions the arithmetic varieties X/Y behave like Galois extensions E/F of number fields; at the same time, their automorphism groups $\text{Aut}(X/Y)$ behave like the Galois groups $\text{Gal}(E/F)$. In particular, the related data of varieties, such as the arithmetic fundamental groups, encode plenty of information of the maximal abelian class fields of the number fields. It needs to decode them for one to obtain class fields.

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Moreover, in such a case, one says that the arithmetic varieties X/Y are a *geometric model* for the Galois extensions E/F if the Galois group $\text{Gal}(E/F)$ is isomorphic to the automorphism group $\text{Aut}(X/Y)$ (for example, see [5, 10, 11, 13, 14]).

Now let F be a finitely generated extension over a number field. In this paper we will have a try to use the related data of arithmetic varieties X/Y to describe a prescribed field E , a finitely generated extension over F , of transcendental degree not less than one. We will give an explicit construction of such a geometric model for function fields in several variables (see *Main Theorem*).

On the other hand, we will also demonstrate the existence of quasi-galois closed covers in [2], which is as a by-product of the procedure for the proof of the *Main Theorem* (see *Theorem 3.4*). It can be regarded as a generalization of the pseudo-galois covers of arithmetic varieties in the sense of Suslin-Voevodsky (see [13, 14]).

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1. STATEMENT OF THE MAIN THEOREM

In the present paper, an **arithmetic variety** is an integral scheme of finite type over $\text{Spec}(\mathbb{Z})$. Let $k(X) \triangleq \mathcal{O}_{X,\xi}$ denote the function field of an arithmetic variety X with generic point ξ .

Let E be a finitely generated extension of a field F . Here E is not necessarily algebraic over F . Then E is said to be a **Galois extension** of F if F is the fixed subfield of the Galois group $\text{Gal}(E/F)$ in E .

The following is the main theorem of the paper.

Theorem 1.1. (The Main Theorem) *Let K be a finitely generated extensions over a number field. Suppose that Y is an arithmetic variety with $K = k(Y)$. Take any finitely generated extensions L of K such that L is Galois over K .*

Then there exists an arithmetic variety X and a surjective morphism $f : X \rightarrow Y$ of finite type such that

- $L = k(X)$;
- the morphism f is affine;
- there is a group isomorphism $\text{Aut}(X/Y) \cong \text{Gal}(L/K)$.

Remark 1.2. Let $\dim X = \dim Y$. Then X/Y are said to be a **geometric model** of the field extension E/F provided that $k(X) = E$ and $k(Y) = F$ and there is a group isomorphism $\text{Aut}(X/Y) \cong \text{Gal}(E/F)$ (for example, see [10, 11, 13, 14]). In the paper *Theorem 1.1* above

gives us a geometric model for function fields in several variables, which is an analogue of the case for finite extensions of number fields.

Remark 1.3. In the course of the proof of *Theorem 1.1*, as a by-product, we will also demonstrate the existence of quasi-galois closed covers in [2] (see *Theorem 3.4* in the paper), which can be regarded as a generalization of the pseudo-galois covers of arithmetic varieties in the sense of Suslin-Voevodsky (see [13, 14]).

2. AN EXPLICIT CONSTRUCTION FOR THE MODEL

2.1. Notation. Let us fix some notation and definitions before we give the procedure for the construction (for details, see [1, 2]). Given an integral domain D . Let $Fr(D)$ denote the field of fractions on D . In particular, if D is a subring of a field Ω , the field $Fr(D)$ will always assumed to be contained in Ω .

Let (X, \mathcal{O}_X) be a scheme. As usual, an affine covering of the scheme (X, \mathcal{O}_X) is a family $\mathcal{C}_X = \{(U_\alpha, \phi_\alpha; A_\alpha)\}_{\alpha \in \Delta}$ such that for each $\alpha \in \Delta$, ϕ_α is an isomorphism from an open set U_α of X onto the spectrum $Spec A_\alpha$ of a commutative ring A_α . Each $(U_\alpha, \phi_\alpha; A_\alpha) \in \mathcal{C}_X$ is called a **local chart**. An affine covering \mathcal{C}_X of (X, \mathcal{O}_X) is said to be **reduced** if $U_\alpha \neq U_\beta$ holds for any $\alpha \neq \beta$ in Δ .

Let **Comm** be the category of commutative rings with identity. Fixed a subcategory **Comm**₀ of **Comm**. An affine covering $\{(U_\alpha, \phi_\alpha; A_\alpha)\}_{\alpha \in \Delta}$ of (X, \mathcal{O}_X) is said to be **with values** in **Comm**₀ if for each $\alpha \in \Delta$ there are $\mathcal{O}_X(U_\alpha) = A_\alpha$ and $U_\alpha = Spec(A_\alpha)$, where A_α is a ring contained in **Comm**₀.

Let Ω be a field and let **Comm**(Ω) be the category consisting of the subrings of Ω and their isomorphisms. An affine covering \mathcal{C}_X of (X, \mathcal{O}_X) with values in **Comm**(Ω) is said to be **with values in the field** Ω .

2.2. Process of the Construction. The following is the procedure for the construction of the geometric model.

Let K be a finitely generated extensions over a number field and let Y be an arithmetic variety such that $K = k(Y)$. Take any finitely generated extensions L of K such that L/K is a Galois extension.

We will proceed in several steps to construct an arithmetic variety X and a surjective morphism $f : X \rightarrow Y$ satisfying the desired property in the *Main Theorem* of the paper, which will be proved in next section.

Step 1. Fixed an algebraic closure Ω_L of L . Put

$$\Omega_K = \Omega_L \cap \overline{K},$$

i.e., an algebraic closure of K .

Without loss of generality, assume that the ring $\mathcal{O}_Y(V)$ is contained in Ω_K for each affine open set V of the scheme Y .

Otherwise, if that property does not hold, by discussion in [1] we can choose a scheme $(Y', \mathcal{O}_{Y'})$ which has that property and is isomorphic to (Y, \mathcal{O}_Y) .

Evidently, that property holds automatically if Y is an affine scheme. Choose the elements

$$t_1, t_2, \dots, t_n \in L \setminus K$$

to be a nice basis of L over K (see [2]), that is, they satisfy the following conditions:

- (i) $L = K(t_1, t_2, \dots, t_n)$;
- (ii) t_1, t_2, \dots, t_r constitute a transcendental basis of L over K ;
- (iii) $t_{r+1}, t_{r+2}, \dots, t_n$ are linearly independent over $K(w_1, w_2, \dots, w_r)$, where $0 \leq r \leq n$.

Let \mathcal{C}_Y be the maximal element (by set inclusion) in the collection of the reduced affine coverings of the scheme Y with values in Ω_K .

Step 2. Take any local chart $(V, \psi_V, B_V) \in \mathcal{C}_Y$. Then V is an affine open subset of Y and we have

$$Fr(B_V) = K \text{ and } \mathcal{O}_Y(V) = B_V \subseteq \Omega_K.$$

Define A_V to be the subring of L generated over B_V by the set of elements in L

$$\Delta_V \triangleq \{\sigma(t_j) \in L : \sigma \in Gal(L/K), 1 \leq j \leq n\}.$$

That is, we have

$$A_V = B_V[\Delta_V].$$

Put

$$\Delta'_V = \Delta_V \setminus \{t_1, t_2, \dots, t_r\}.$$

We have

$$Fr(A_V) = L;$$

$$A_V = B_V[t_1, t_2, \dots, t_r][\Delta'_V].$$

Then Δ'_V is a nonvoid set. It is seen that B_V is exactly the invariant subring of the natural action of the Galois group $Gal(L/K)$ on A_V .

Set

$$i_V : B_V \rightarrow A_V$$

to be the inclusion.

Step 3. Define the disjoint union

$$\Sigma = \coprod_{(V, \psi_V, B_V) \in \mathcal{C}_Y} Spec(A_V).$$

Let

$$\pi_Y : \Sigma \rightarrow Y$$

be the projection.

Σ is a topological space, where the topology τ_Σ on Σ is naturally determined by the Zariski topologies on all $\text{Spec}(A_V)$.

Step 4. Define an equivalence relation R_Σ in Σ in such a manner:

Take any $x_1, x_2 \in \Sigma$. We say

$$x_1 \sim x_2$$

if and only if

$$j_{x_1} = j_{x_2}$$

holds in L .

Here, j_x denotes the corresponding prime ideal of A_V to a point $x \in \text{Spec}(A_V)$ (see [4]).

Define

$$X = \Sigma / \sim .$$

Let

$$\pi_X : \Sigma \rightarrow X$$

be the projection.

Hence, X is a topological space as a quotient of Σ .

Step 5. Define a map

$$f : X \rightarrow Y$$

by

$$\pi_X(z) \longmapsto \pi_Y(z)$$

for each $z \in \Sigma$.

Step 6. Put

$$\mathcal{C}_X = \{(U_V, \varphi_V, A_V)\}_{(V, \psi_V, B_V) \in \mathcal{C}_Y}$$

where $U_V = \pi_Y^{-1}(V)$ holds and $\varphi_V : U_V \rightarrow \text{Spec}(A_V)$ is the identity map for each $(V, \psi_V, B_V) \in \mathcal{C}_Y$. Then \mathcal{C}_X is a reduced affine covering on the space X with values in Ω_L .

Define the scheme

$$(X, \mathcal{O}_X)$$

to be obtained by gluing the affine schemes $\text{Spec}(A_V)$ for all local charts $(V, \psi_V, B_V) \in \mathcal{C}_Y$ with respect to the equivalence relation R_Σ (see [4, 6]).

Then \mathcal{C}_X is admissible and the sheaf \mathcal{O}_X is an extension of \mathcal{C}_X on the space X (see [1]).

Finally, (X, \mathcal{O}_X) is the desired scheme and $f : X \rightarrow Y$ is the desired morphism of schemes. (Note that the proof will be given in the following section.)

This completes the construction.

3. PROOF OF THE MAIN THEOREM

3.1. Definitions. Assume that \mathcal{O}_X and \mathcal{O}'_X are two structure sheaves on the underlying space of an integral scheme X . The integral schemes (X, \mathcal{O}_X) and (X, \mathcal{O}'_X) are said to be **essentially equal** provided that for any open set U in X , we have

$$U \text{ is affine open in } (X, \mathcal{O}_X) \iff \text{so is } U \text{ in } (X, \mathcal{O}'_X)$$

and in such a case, $D_1 = D_2$ holds or there is $Fr(D_1) = Fr(D_2)$ such that for any nonzero $x \in Fr(D_1)$, either

$$x \in D_1 \bigcap D_2$$

or

$$x \in D_1 \setminus D_2 \iff x^{-1} \in D_2 \setminus D_1$$

holds, where $D_1 = \mathcal{O}_X(U)$ and $D_2 = \mathcal{O}'_X(U)$.

Two schemes (X, \mathcal{O}_X) and (Z, \mathcal{O}_Z) are said to be **essentially equal** if the underlying spaces of X and Z are equal and the schemes (X, \mathcal{O}_X) and (X, \mathcal{O}_Z) are essentially equal.

Let X and Y be two arithmetic varieties and let $f : X \rightarrow Y$ be a surjective morphism of finite type. By a **conjugate** Z of X over Y we understand an arithmetic variety Z that is isomorphic to X over Y . Let $Aut(X/Y)$ denote the group of automorphisms of X over Y .

Then X is said to be **quasi-galois closed** over Y by f if there is an algebraically closed field Ω and a reduced affine covering \mathcal{C}_X of X with values in Ω such that for any conjugate Z of X over Y the two conditions are satisfied:

- (X, \mathcal{O}_X) and (Z, \mathcal{O}_Z) are essentially equal if Z has a reduced affine covering with values in Ω .
- $\mathcal{C}_Z \subseteq \mathcal{C}_X$ holds if \mathcal{C}_Z is a reduced affine covering of Z with values in Ω .

Let K be an extension of a field k . Here K/k is not necessarily algebraic. Recall that K is said to be **quasi-galois** over k if each irreducible polynomial $f(X) \in F[X]$ that has a root in K factors completely in $K[X]$ into linear factors for any intermediate field $k \subseteq F \subseteq K$ (see [2]).

The elements

$$t_1, t_2, \dots, t_n \in K \setminus k$$

to be a **nice basis** of K over k if they satisfy the following conditions:

- (i) $L = K(t_1, t_2, \dots, t_n)$;
- (ii) t_1, t_2, \dots, t_r constitute a transcendental basis of L over K ;

(iii) $t_{r+1}, t_{r+2}, \dots, t_n$ are linearly independent over $K(t_1, t_2, \dots, t_r)$, where $0 \leq r \leq n$.

Now let $D \subseteq D_1 \cap D_2$ be three integral domains. The ring D_1 is said to be **quasi-galois** over D if the field $Fr(D_1)$ is a quasi-galois extension of $Fr(D)$.

The ring D_1 is said to be a **conjugation** of D_2 over D if there is a (r, n) -nice k -basis w_1, w_2, \dots, w_n of the field $Fr(D_1)$ and an F -isomorphism $\tau_{(r,n)} : Fr(D_1) \rightarrow Fr(D_2)$ of fields such that

$$\tau_{(r,n)}(D_1) = D_2,$$

where $k = Fr(D)$ and $F \triangleq k(w_1, w_2, \dots, w_r)$ is assumed to be contained in the intersection $Fr(D_1) \cap Fr(D_2)$.

3.2. Criterion for Quasi-galois Closed. Let X and Y be two arithmetic varieties. Let Ω be a fixed algebraically closed closure of the function field $k(X)$.

Definition 3.1. Let $\varphi : X \rightarrow Y$ be a surjective morphism of finite type. A reduced affine covering \mathcal{C}_X of X with values in Ω is said to be **quasi-galois closed** over Y by φ if the below condition is satisfied:

There exists a local chart $(U'_\alpha, \phi'_\alpha; A'_\alpha) \in \mathcal{C}_X$ such that $U'_\alpha \subseteq \varphi^{-1}(V_\alpha)$ for any $(U_\alpha, \phi_\alpha; A_\alpha) \in \mathcal{C}_X$, for any affine open set V_α in Y with $U_\alpha \subseteq \varphi^{-1}(V_\alpha)$, and for any conjugate A'_α of A_α over B_α , where B_α is the canonical image of $\mathcal{O}_Y(V_\alpha)$ in the function field $k(Y)$.

Lemma 3.2. *Let $\varphi : X \rightarrow Y$ be a surjective morphism of finite type. Suppose that the function field $k(Y)$ is contained in Ω . Then the scheme X is quasi-galois closed over Y if there is a unique maximal reduced affine covering \mathcal{C}_X of X with values in Ω such that \mathcal{C}_X is quasi-galois closed over Y .*

Proof. Assume that there is a unique maximal reduced affine covering \mathcal{C}_X of X with values in Ω such that \mathcal{C}_X is quasi-galois closed over Y .

Fixed any a conjugate Z of X over Y . Let $\sigma : Z \rightarrow X$ be an isomorphism of schemes over Y . Suppose that Z has a reduced affine covering \mathcal{C}_Z with values in Ω .

Take any local chart $(W, \delta, C) \in \mathcal{C}_Z$. Put

$$\begin{aligned} U &= \sigma(W); \\ A &= \mathcal{O}_X(U); \\ C &= \mathcal{O}_Z(W). \end{aligned}$$

Then we have

$$U = \text{Spec}(A) \text{ and } W = \text{Spec}(C).$$

As \mathcal{C}_X is quasi-galois closed over Y , it is seen that there is an affine open subset U' in X such that

$$C = \mathcal{O}_X(U').$$

As

$$U' = \text{Spec}(C) = W,$$

we have

$$\sigma^{-1}(U) = U' \subseteq X;$$

hence,

$$Z = \sigma^{-1}(X) = X.$$

It follows that we must have $(X, \mathcal{O}_X) = (Z, \mathcal{O}_Z)$. \square

An affine covering $\{(U_\alpha, \phi_\alpha; A_\alpha)\}_{\alpha \in \Delta}$ of (X, \mathcal{O}_X) is said to be an **affine patching** of (X, \mathcal{O}_X) if ϕ_α is the identity map on $U_\alpha = \text{Spec} A_\alpha$ for each $\alpha \in \Delta$.

Evidently, an affine patching is reduced.

Lemma 3.3. *Let $\varphi : X \rightarrow Y$ be a surjective morphism of finite type. Suppose that the function field $k(Y)$ is contained in Ω . Then X is quasi-galois closed over Y if there is a unique maximal affine patching \mathcal{C}_X of X with values in Ω such that*

- *either \mathcal{C}_X is quasi-galois closed over Y ,*
- *or A_α has only one conjugate over B_α for any $(U_\alpha, \phi_\alpha; A_\alpha) \in \mathcal{C}_X$ and for any affine open set V_α in Y with $U_\alpha \subseteq \varphi^{-1}(V_\alpha)$, where B_α is the canonical image of $\mathcal{O}_Y(V_\alpha)$ in the function field $k(Y)$.*

Proof. It is immediate from Lemma 3.2. \square

3.3. Existence of Quasi-galois Closed Covers. Now we give the existence of quasi-galois closed covers which take values in a prescribed extension of the function field in several variables.

Theorem 3.4. *Let K be a finitely generated extensions of a number field and let Y be an arithmetic variety with $K = k(Y)$. Fixed any finitely generated extensions L of K such that L is Galois over K .*

Then there exists an arithmetic variety X and a surjective morphism $f : X \rightarrow Y$ of finite type such that

- *$L = k(X)$;*
- *the morphism f is affine;*
- *X is a quasi-galois closed over Y by f .*

Proof. It is immediate from Lemma 3.3 and the construction in §2. \square

3.4. Proof of the Main Theorem. Now we can give the proof of the Main Theorem of the paper.

Proof. (Proof of Theorem 1.1) It is immediate from *Theorem 3.4* above and the *Main Theorem* in [2]. \square

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